Title	A note on equivalence relations of pair of germs (Theory of singularities of smooth mappings and around it)
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Citation	数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2016), B55: 17-39
Issue Date	2016-04
URL	http://hdl.handle.net/2433/241304
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Type	Departmental Bulletin Paper
Textversion	publisher

RIMS Kôkyûroku Bessatsu
B55 (2016), 17–39

A note on equivalence relations of pair of germs

By

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Abstract

In this paper we recover the definition of bi-$\mathcal{K}$-equivalence and give a concise overview of this theory. After this, we introduce the notion of topological bi-$\mathcal{K}$-equivalence showing some examples and properties. In the last part of the paper, some open questions about this topic are proposed.

§ 1. Introduction

It is classical in Singularity theory that $\mathcal{A}$-classification of stable map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ can be reduced to $\mathcal{K}$-classification which means classification of isomorphic $\mathbb{R}$-algebras (cf. [15], [16]). Our purpose here is to provide relationships between other versions of $\mathcal{A}$ and $\mathcal{K}$ equivalences, adapted for pairs of germs of type $(f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$. A pair of such germs can be seen as a divergent diagram $(\mathbb{R}^q, 0) \xleftarrow{f_2} (\mathbb{R}^n, 0) \xrightarrow{f_1} (\mathbb{R}^p, 0)$. Divergent diagrams have many applications. For instance, in envelope theory, web geometry, singularities of first order differential equations and vision theory. This subject was discussed by several authors including V. Arnold, J.P. Dufour, M.A. Teixeira and M. Ruas (see for instance, [2], [9], [10], [11], [24], [13]).

This paper is divided in two parts: in the first part (Sections 2-4), we present the definition of bi-$\mathcal{K}$-equivalence introduced by L. Favaro and his students in decade of 80

Received June 27, 2014. Revised October 11, 2014.
2010 Mathematics Subject Classification(s): 58K15, 58K40
Key Words: *bi-$\mathcal{K}$-equivalence, classification, divergent diagrams.*
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(cf. [21], [22], [23], [20]) motivated by Dufour's work about bi-$\mathcal{A}$-equivalence of couples of germs (cf. [9], [11]). We give a concise overview of this theory based in [21], [19], [20]. Our motivation is to recover this subject and to prove some results which are incomplete or partially proved in those previous works. In the second part (Section 5), we introduce the notion of topological bi-$\mathcal{K}$-equivalence. Some properties and examples are given. We notice that recently several papers have considered the topological case (cf. [17], [18], [5], [1], [4], [6], among others). Since this is an article to promote this subject, some open questions are proposed in the last part of the paper.

§ 2. Classical results: $\mathcal{A}$ and $\mathcal{K}$-equivalences

Denote by $\varepsilon_{n,p}$ the set of all smooth map germs $\{f : (\mathbb{R}^n, 0) \to \mathbb{R}^p \mid f \in C^\infty\}$. When $p = 1$, denote $\varepsilon_{n,1}$ just by ε_n which is a local ring. The unique maximal ideal of ε_n is denoted by $\mathcal{M}_n$ which consists of all germs f such that $f(0) = 0$.

Denote by $\varepsilon^o_{n,p} = \{f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \mid f \in C^\infty\}$ and by

$$Q(f) = \varepsilon_n / I_f$$

the local $\mathbb{R}$-algebra associated to f, where I_f is the ideal of ε_n generated by components $(f_1, \ldots, f_p)$ of f. The jacobian ideal of f is denoted by Jf and the notation $\langle \cdot \rangle_{\varepsilon_n}$ indicates an ideal in ε_n.

The most natural question about classification of map germs in Singularity theory is to decide if two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are *$\mathcal{A}$-equivalent*. That is, if there exist C^∞ diffeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that

$$g = k \circ f \circ h.$$

However such classification is hard. For some special classes of map germs it is possible to obtain a reasonable answer to the $\mathcal{A}$-classification problem. For instance, for stable map germs we have an answer due to J. Mather (cf. [15], [16]). In order, Mather reduced the $\mathcal{A}$-classification problem of stable map gems to classification problem of isomorphic $\mathbb{R}$-algebras, introducing the notion of contact equivalence (or $\mathcal{K}$-equivalence).

Definition 2.1. Two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be *$\mathcal{K}$-equivalent* if there exist two germs of C^∞ diffeomorphisms

$$H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0) \quad \text{and} \quad h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{R}^n, 0) & \xrightarrow{(id_n, f)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\ h \downarrow & & H \downarrow & & h \downarrow \\ (\mathbb{R}^n, 0) & \xrightarrow{(id_n, g)} & (\mathbb{R}^n \times \mathbb{R}^p, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \end{array}$$

where id_n is the identity map germ of $\mathbb{R}^n$ and π_n is the canonical projection germ.

When $h = id_n$, we say that f and g are $\mathcal{C}$-equivalent.

To investigate recognition problem, a key notion in Singularity theory is the finite determinacy. If f is finitely determined, then we may assume that f is polynomial.

Definition 2.2. Let G be any equivalence relation between map germs. We say that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is k-G-determined if for any map germ g with $j^k g(0) = j^k f(0)$, g is G-equivalent to f. We say that f is finitely G-determined if it is k-G-determined for some k. Here $j^k f(0)$ denotes the k-jet of f at 0.

Notice that if f is finitely k-G-determined then f is G-equivalent to $j^k f(0)$, which is polynomial. For map germs which are finitely $\mathcal{K}$-determined, the notion of $\mathcal{K}$-equivalence has an algebraic characterization:

Theorem 2.3. *(Mather [16]) Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be finitely $\mathcal{K}$-determined smooth map germs. Then f is $\mathcal{K}$-equivalent to g if and only if their local $\mathbb{R}$-algebras $Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras.*

The next classification theorem shows that for stable map germs the notions of $\mathcal{A}$ and $\mathcal{K}$ equivalences are equivalent:

Theorem 2.4. *(Mather [16]) Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be stable map germs. Then f is $\mathcal{A}$-equivalent to g if and only if they are $\mathcal{K}$-equivalent.*

There are other algebraic characterization for $\mathcal{K}$-equivalence which makes it fairly computable notion. We show this in next two propositions:

Proposition 2.5. *([12]) Let $f, g \in \varepsilon^o_{n,p}$. The following conditions are equivalent:*

i) f and g are $\mathcal{C}$-equivalent;

ii) The ideals I_f and I_g are equal;

iii) There exists an invertible matrix $p \times p$, (u_{ij}), with $u_{ij} \in \varepsilon_n$, such that

$$f_i = \sum_j u_{ij} g_j, \quad 1 \le i \le p.$$

Proposition 2.6. *([12]) Two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are $\mathcal{K}$-equivalent if and only if there exists a germ of C^∞ diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that g and $f \circ h$ are $\mathcal{C}$-equivalent.*

Obviously via Proposition 2.6 there exist similar conditions of Proposition 2.5 for $\mathcal{K}$-equivalence of two map germs.

§ 3. Equivalence relations of pair of germs

In [9], J.P. Dufour introduced the notion of bi-stability for a couple of germs $(f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ and studied the classification problem of these pairs in some particular cases. Motivated by Dufour's work, L.A. Favaro and his students in decade 80 introduced the notion of bi-$\mathcal{K}$-equivalence to study the bi-$\mathcal{A}$-equivalence of such pairs, similar to the approach taken by Mather.

Observe that a pair of germs $(f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ can be seen as a divergent diagram

$$(\mathbb{R}^q, 0) \xleftarrow{f_2} (\mathbb{R}^n, 0) \xrightarrow{f_1} (\mathbb{R}^p, 0).$$

Definition 3.1. Two smooth pairs of germs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are said to be *bi-$\mathcal{A}$-equivalent* if there exist C^∞ diffeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $k_1 : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ and $k_2 : (\mathbb{R}^q, 0) \to (\mathbb{R}^q, 0)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{R}^n, 0) & \xrightarrow{(f_1, f_2)} & (\mathbb{R}^p \times \mathbb{R}^q, 0) \\ h \downarrow & & \downarrow k_1 \times k_2 \\ (\mathbb{R}^n, 0) & \xrightarrow{(g_1, g_2)} & (\mathbb{R}^p \times \mathbb{R}^q, 0) \end{array}$$

Remark. 1. In a similar way, we can obtain the definition of bi-$\mathcal{A}$-equivalence for two divergent diagrams $(\mathbb{R}^q, 0) \xleftarrow{f_2} (\mathbb{R}^n, 0) \xrightarrow{f_1} (\mathbb{R}^p, 0)$ and $(\mathbb{R}^q, 0) \xleftarrow{g_2} (\mathbb{R}^n, 0) \xrightarrow{g_1} (\mathbb{R}^p, 0)$.

The diagram in the Definition 3.1 can be rewritten as

$$\begin{array}{ccccc} (\mathbb{R}^q, 0) & \xleftarrow{f_2} & (\mathbb{R}^n, 0) & \xrightarrow{f_1} & (\mathbb{R}^p, 0) \\ k_2 \downarrow & & \downarrow h & & \downarrow k_1 \\ (\mathbb{R}^q, 0) & \xleftarrow{g_2} & (\mathbb{R}^n, 0) & \xrightarrow{g_1} & (\mathbb{R}^p, 0) \end{array}$$

2. If (f_1, f_2) and (g_1, g_2) are bi-$\mathcal{A}$-equivalent, then $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are $\mathcal{A}$-equivalent.

3. If (f_1, f_2) and (g_1, g_2) are bi-$\mathcal{A}$-equivalent, then f_1 and g_1 are $\mathcal{A}$-equivalent and also f_2 and g_2 are $\mathcal{A}$-equivalent.

Question 1. How to describe the classification of pairs of germs (or divergent diagrams) with respect to the bi-$\mathcal{A}$-equivalence?

This is an important question because divergent diagrams of map germs appear in several geometrical contexts. In [9] a generic classification of divergent diagrams of type

$(\mathbb{R},0) \xleftarrow{f_2} (\mathbb{R}^2,0) \xrightarrow{f_1} (\mathbb{R}^2,0)$ was obtained. We can find other classifications involving divergent diagrams (cf. [2], [9], [10], [11], [24], [13] among others).

In this paper we investigate the Question 1 for bi-stable pairs of germs. The answer for Question 1 is given in terms of bi-$\mathcal{K}$-equivalence of such pairs. The results described here are also contained in [21], [20] or [19]. However, some of them are not completely proved in those previous works. Here we recover them and give a complete proof.

Definition 3.2. Two pairs of smooth map germs $(f_1,f_2),(g_1,g_2) : (\mathbb{R}^n,0) \to (\mathbb{R}^p \times \mathbb{R}^q,0)$ are said to be *bi-$\mathcal{K}$-equivalent* if there exist C^∞ diffeomorphisms

$$H : (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q,0) \to (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q,0) \quad \text{and} \quad h : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$$

such that the following diagram is commutative:

$$\begin{array}{ccccc}
(\mathbb{R}^n,0) & \xrightarrow{(id_n,(f_1,f_2))} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q,0) & \xrightarrow{\pi_n} & (\mathbb{R}^n,0) \\
h \downarrow & & H \downarrow & & h \downarrow \\
(\mathbb{R}^n,0) & \xrightarrow{(id_n,(g_1,g_2))} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q,0) & \xrightarrow{\pi_n} & (\mathbb{R}^n,0)
\end{array}$$

and for which we have $H = (h \circ \pi_n, H_1 \circ \pi_{n,p}, H_2 \circ \pi_{n,q})$ where id_n is the identity map germ of $\mathbb{R}^n$; π_n is the usual projection in $\mathbb{R}^n$; $\pi_{n,p}$ is the usual projection in $\mathbb{R}^n \times \mathbb{R}^p$; $\pi_{n,q}$ is the usual projection in $\mathbb{R}^n \times \mathbb{R}^q$; $H_1 : (\mathbb{R}^n \times \mathbb{R}^p,0) \to (\mathbb{R}^n \times \mathbb{R}^p,0)$ and $H_2 : (\mathbb{R}^n \times \mathbb{R}^q,0) \to (\mathbb{R}^n \times \mathbb{R}^q,0)$.

When $h = id_n$, we say that (f_1,f_2) and (g_1,g_2) are bi-$\mathcal{C}$-equivalent.

Remark. 1. From commutativity of diagram in Definition 3.2 we can write

$$H(x,y,z) = (h(x), H_1(x,y), H_2(x,z)), \quad H_1(x,0) = H_2(x,0) = 0, \quad \forall x \in (\mathbb{R}^n,0).$$

2. If (f_1,f_2) and (g_1,g_2) are bi-$\mathcal{K}$-equivalent, then $f = (f_1,f_2)$ and $g = (g_1,g_2)$ are $\mathcal{K}$-equivalent.

3. If (f_1,f_2) and (g_1,g_2) are bi-$\mathcal{K}$-equivalent, the diagram in Definition 3.2 can be written as the following commutative diagram

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^q,0) & \xleftarrow{(id_n,f_2)} & (\mathbb{R}^n,0) & \xrightarrow{(id_n,f_1)} & (\mathbb{R}^n \times \mathbb{R}^p,0) \\
H_2 \downarrow & & h \downarrow & & H_1 \downarrow \\
(\mathbb{R}^n \times \mathbb{R}^q,0) & \xleftarrow{(id_n,g_2)} & (\mathbb{R}^n,0) & \xrightarrow{(id_n,g_1)} & (\mathbb{R}^n \times \mathbb{R}^p,0)
\end{array}$$

In other words, if the pairs of germs (f_1,f_2) and (g_1,g_2) are bi-$\mathcal{K}$-equivalent, then the germs f_1 and g_1 are $\mathcal{K}$-equivalent and also f_2 and g_2 are $\mathcal{K}$-equivalent.

4. It is easy to see that bi-$\mathcal{A}$-equivalence implies bi-$\mathcal{K}$-equivalence.

Similar to classical Singularity theory, it is possible to adapt many classical results involving contact equivalence to obtain new results for bi-$\mathcal{K}$-equivalence. First this was done by E.A. Silva and by the second named author in their thesis ([21], [19]). Here we recover some of these results:

Proposition 3.3. *([21],[19]) The pairs of germs (f_1, f_2), $(g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-$\mathcal{C}$-equivalent if and only if $I_{f_1} = I_{g_1}$ and $I_{f_2} = I_{g_2}$.*

Proposition 3.4. *([21],[19]) The pairs of germs (f_1, f_2), $(g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-$\mathcal{K}$-equivalent if and only if there exists a C^∞ diffeomorphism map germ $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that (f_1, f_2) and $(g_1, g_2) \circ h$ are bi-$\mathcal{C}$-equivalent.*

Proposition 3.5. *([21],[19]) The pairs of germs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-$\mathcal{C}$-equivalent if and only if there exist invertible matrices $p \times p$ and $q \times q$, (u_{ij}) and (v_{rs}), with $u_{ij}, v_{rs} \in \varepsilon_n$, such that*

$$f_{1i} = \sum_j u_{ij} g_{1j}, \quad i = 1, \ldots, p,\ j = 1, \ldots, p; \quad \text{and}$$

$$f_{2r} = \sum_s v_{rs} g_{2s}, \quad r = 1, \ldots, q,\ s = 1, \ldots, q,$$

where $f_1 = (f_{11}, \ldots, f_{1p}), g_1 = (g_{11}, \ldots, g_{1p})$ and $f_2 = (f_{21}, \ldots, f_{2q}), g_2 = (g_{21}, \ldots, g_{2q})$.

Proposition 3.6. *([21],[19]) The pairs of germs (f_1, f_2), $(g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-$\mathcal{K}$-equivalent if and only if I_{f_1}, I_{g_1} and I_{f_2},I_{g_2} are induced isomorphic by the same germ of diffeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ (that is, $I_{f_1} = h^\star(I_{g_1})$ and $I_{f_2} = h^\star(I_{g_2})$).*

Using the previous results is easy to check the following examples:

Example 3.7. The pairs $(f_1, f_2) = (x^3 - x^2, x^2 + x)$ and $(g_1, g_2) = (x^2, x)$ are bi-$\mathcal{C}$-equivalent. Hence they are bi-$\mathcal{K}$-equivalent.

Example 3.8. The pairs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$ given by $(f_1, f_2)(x) = (x^2, 0)$ and $(g_1, g_2)(x) = (x^2, x^3)$ are $\mathcal{C}$-equivalent (hence $\mathcal{K}$-equivalent) but they are not bi-$\mathcal{K}$-equivalent, because $I_{f_1} = \langle x^2 \rangle = I_{g_1}$ and $I_{f_2} = \langle 0 \rangle$ and $I_{g_2} = \langle x^3 \rangle$ are not induced isomorphic by the same diffeomorphism. These pairs of germs also are not bi-$\mathcal{A}$-equivalent (since they are not $\mathcal{A}$-equivalent).

§ 3.1. The bi-$\mathcal{K}$-tangent space and finite determinacy

The bi-$\mathcal{K}$-tangent space is defined in a natural way as following:

Definition 3.9. The *bi-$\mathcal{K}$-tangent space* of a pair of germs $(f_1, f_2) \in \varepsilon^0_{n,p+q}$ is the ε_n-submodule of $\varepsilon^0_{n,p+q}$ given by

$$TKK \cdot (f_1, f_2) = J(f_1, f_2) + \{[I_{f_1} \cdot \varepsilon^0_{n,p} \times \{0\}] + [\{0\} \times I_{f_2} \cdot \varepsilon^0_{n,q}]\}.$$

The *bi-$\mathcal{K}$-codimension* of (f_1, f_2) is defined as the codimension of $TKK \cdot (f_1, f_2)$ as a real vector subspace of $\varepsilon^0_{n,p+q}$.

Proposition 3.10. *([21], [19]) The pair $(f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ has finite bi-$\mathcal{K}$-codimension if and only if there exists $\ell \in \mathbb{Z}$, $\ell \geq 1$, such that $\mathcal{M}^\ell_n \cdot \varepsilon^0_{n,p+q} \subset TKK \cdot (f_1, f_2)$.*

Proposition 3.11. *([21], [19]) If two pairs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-$\mathcal{K}$-equivalent then they have the same bi-$\mathcal{K}$-codimension.*

Of course the bi-$\mathcal{K}$-tangent space of (f_1, f_2) is a vector subspace of the classical $\mathcal{K}$-tangent space of $f = (f_1, f_2)$. Then,

$$\text{bi-}\mathcal{K}\text{-codimension } (f_1, f_2) \geq \mathcal{K}\text{-codimension of } f, \qquad f = (f_1, f_2).$$

Given $(f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$, is easy to check the following inequality:

$$\mathcal{K}\text{-cod } (f_1) + \mathcal{K}\text{-cod } (f_2) \leq 2 \cdot \text{bi-}\mathcal{K}\text{-cod } (f_1, f_2).$$

In fact, the previous inequality can be improved as following (see [21], [19]):

$$\mathcal{K}\text{-cod } (f_1) + \mathcal{K}\text{-cod } (f_2) \leq \text{bi-}\mathcal{K}\text{-cod } (f_1, f_2).$$

Example 3.12. Consider the pairs of germs $(f_1, f_2)(x) = (x^2, 0)$ and $(g_1, g_2)(x) = (x^2, x^3)$. In this case,

$$TKK \cdot (f_1, f_2) = \langle (x, 0) \rangle$$

then the bi-$\mathcal{K}$-cod $(f_1, f_2) = +\infty$. By other hand,

$$TKK \cdot (g_1, g_2) = \langle (2x, 3x^2), (x^2, 0), (0, x^3) \rangle.$$

Hence the bi-$\mathcal{K}$-cod $(g_1, g_2) = 4$.

By Proposition 3.11, the pairs (f_1, f_2) and (g_1, g_2) are not bi-$\mathcal{K}$-equivalent. However, by Example 3.8 the germs $f = (x^2, 0)$ and $g = (x^2, x^3)$ are $\mathcal{K}$-equivalent. Then, f and g have the same $\mathcal{K}$-codimension (cf. [12]).

Example 3.13. The pairs $(f_1, f_2)(x) = (x^3 - x^2, x^2 + x)$ and $(g_1, g_2)(x) = (x^2, x)$ given in Example 3.7 have the respective bi-$\mathcal{K}$-tangent spaces:

$$TKK \cdot (f_1, f_2) = \langle (3x^2 - 2x, 2x + 1), (x^3 - x^2, 0), (0, x^2 + x) \rangle$$

and

$$TKK \cdot (g_1, g_2) = \langle (2x, 1), (x^2, 0), (0, x) \rangle.$$

In this case, both pairs of germs have bi-$\mathcal{K}$-codimension 2.

§ 3.2. Bi-$\mathcal{K}$-versal deformations of pairs

Definition 3.14. An *r-deformation of a pair* $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ is a pair of germs $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ for which $(f_1, f_2)(0, x) = (f_{01}, f_{02})(x)$.

Definition 3.15. Two pairs of r-deformations $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ of (f_{01}, f_{02}) are said to be *bi-$\mathcal{K}$-equivalent as deformation* if they are bi-$\mathcal{K}$-equivalent as a pair of germs.

Let (f_1, f_2) be an r-deformation of (f_{01}, f_{02}) and consider $\gamma : (\mathbb{R}^s, 0) \to (\mathbb{R}^r, 0)$ be a C^∞ map germ. We define the *pull back of* (f_1, f_2) *by* γ as the following map germ

$$\gamma^\star(f_1, f_2)(v, x) = (f_1(\gamma(v), x), f_2(\gamma(v), x)).$$

The map germ γ is called a *change of parameters* and $\gamma^\star(f_1, f_2) = (\gamma^\star f_1, \gamma^\star f_2)$ is called *deformation induced by* γ.

Definition 3.16. Two pairs of r-deformations $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ of (f_{01}, f_{02}) are said to be *bi-$\mathcal{K}$-isomorphic* if there exists a diffeomorphism germ $\gamma : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ such that (g_1, g_2) is bi-$\mathcal{K}$-equivalent to $\gamma^\star(f_1, f_2)$.

If we consider two pairs of deformations of (f_{01}, f_{02}) with r and s parameters, respectively, then we can compare them via the change of parameters $\gamma : (\mathbb{R}^s, 0) \to (\mathbb{R}^r, 0)$. In fact, if $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ and $(g_1, g_2) : (\mathbb{R}^s \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are such deformations of (f_{01}, f_{02}), we have that $\gamma^\star(f_1, f_2)$ and (g_1, g_2) have the same number of parameters.

Definition 3.17. An r-deformation (f_1, f_2) of (f_{01}, f_{02}) is said to be *bi-$\mathcal{K}$-versal* if any other s-deformation (g_1, g_2) of (f_{01}, f_{02}) is bi-$\mathcal{K}$-equivalent to $\gamma^\star(f_1, f_2)$, for some change of parameters $\gamma : (\mathbb{R}^s, 0) \to (\mathbb{R}^r, 0)$.

If the pair (f_{01}, f_{02}) has bi-$\mathcal{K}$-codimension $= c < \infty$, then (f_1, f_2) as c-parameter bi-$\mathcal{K}$-versal deformation of (f_{01}, f_{02}) is called *bi-$\mathcal{K}$-miniversal deformation.*

Given the pair $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ which is an r-deformation of (f_{01}, f_{02}), we define the initial velocities of (f_1, f_2) by

$$(\dot{f}_1, \dot{f}_2)_i(x) = \left(\frac{\partial f_1}{\partial u_i}(0, x), \frac{\partial f_2}{\partial u_i}(0, x) \right), \quad i = 1, \dots, r.$$

Definition 3.18. An r-deformation (f_1, f_2) of (f_{01}, f_{02}) is said to be *bi-$\mathcal{K}$-transversal* when the vector subspace $\mathbb{R} \cdot \{(\dot{f}_1, \dot{f}_2)_1, \dots, (\dot{f}_1, \dot{f}_2)_r\}$ is satisfied the condition

$$TKK \cdot (f_{01}, f_{02}) + \mathbb{R} \cdot \{(\dot{f}_1, \dot{f}_2)_1, \dots, (\dot{f}_1, \dot{f}_2)_r\} = \varepsilon^0_{n,p+q}.$$

Remark. (f_{01}, f_{02}) admits a bi-$\mathcal{K}$-transversal deformation if and only if it has finite bi-$\mathcal{K}$-codimension.

The next crucial proposition is cited in [21] and partially proved in [19]. Here we will give a complete proof of it in Subsection 3.3.

Proposition 3.19. *A deformation (f_1, f_2) of (f_{01}, f_{02}) is bi-$\mathcal{K}$-versal if and only if it is bi-$\mathcal{K}$-transversal.*

As a consequence, a pair of germs (f_{01}, f_{02}) admits a bi-$\mathcal{K}$-versal deformation if and only if it has finite bi-$\mathcal{K}$-codimension.

Proposition 3.20. *([21], [19]) Let (f_1, f_2) and (g_1, g_2) be two bi-$\mathcal{K}$-miniversal deformations of (f_{01}, f_{02}) and suppose bi-$\mathcal{K}$-cod $(f_{01}, f_{02}) = c$. Then, (f_1, f_2) and (g_1, g_2) are bi-$\mathcal{K}$-isomorphic.*

Let (f_{01}, f_{02}) be a pair of germs with bi-$\mathcal{K}$-codimension c and consider (f_1, f_2) a bi-$\mathcal{K}$-miniversal deformation of (f_{01}, f_{02}). For $d \geq c$, any d-parameter bi-$\mathcal{K}$-versal deformation of (f_{01}, f_{02}) will be bi-$\mathcal{K}$-isomorphic to constant deformation of (f_1, f_2), with $(d - c)$-parameters. Hence, any two d-parameter bi-$\mathcal{K}$-versal deformations of (f_{01}, f_{02}) will be bi-$\mathcal{K}$-isomorphic.

§ 3.3. Proof of Proposition 3.19

To prove the Proposition 3.19 two lemmas are necessary. These lemmas are proved in [21] and [3], respectively. One of them, is an adaptation for pairs of classical Reduction Lemma given by J. Martinet in [14].

Let $(f_1, f_2) : (\mathbb{R}^{r+1} \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be an $(r+1)$-deformation of (f_{01}, f_{02}). Let $(f_1, f_2)^0 : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be the r-deformation given by the restriction of (f_1, f_2) to subspace $u_0 = 0$, i.e.,

$$(f_1, f_2)^0(u_1, \dots, u_r, x) = (f_1, f_2)(0, u_1, \dots, u_r, x).$$

Let $\gamma : (\mathbb{R}^{r+1}, 0) \to (\mathbb{R}^r, 0)$ be any germ and consider the $(r + 1)$-deformation $\gamma^\star (f_1, f_2)^0$ given by

$$\gamma^\star (f_1, f_2)^0 : (\mathbb{R}^{r+1} \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$$

$$(u_0, u_1, \ldots, u_r, x) \mapsto (f_1, f_2)^0(\gamma(u), x) = (f_1, f_2)(0, \gamma(u), x).$$

where $u = (u_0, \ldots, u_r)$.

Lemma 3.21. *(Reduction Lemma [21]) Let X be a germ of vector field at origin in $\mathbb{R}^{r+1} \times \mathbb{R}^n$ given by*

$$X = \frac{\partial}{\partial u_0} + \sum_{i=1}^{r} \xi_i(u) \frac{\partial}{\partial u_i} + \sum_{j=1}^{n} X_j(u, x) \frac{\partial}{\partial x_j}$$

with $X_j(u, 0) = 0$, and X satisfying the following conditions:

$$Df_1 \cdot X \in I_{f_1} \cdot \varepsilon_{r+1+n,p} \quad \text{and} \quad Df_2 \cdot X \in I_{f_2} \cdot \varepsilon_{r+1+n,q}.$$

Then, the pair $(f_1, f_2) : (\mathbb{R}^{r+1} \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ is bi-$\mathcal{K}$-equivalent (as deformation) to the pair $\gamma^\star(f_1, f_2)^0$, where $\gamma : (\mathbb{R}^{r+1}, 0) \to (\mathbb{R}^r, 0)$ is a submersion germ.

Lemma 3.22. *([3]) Let $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be an r-deformation of $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$. Let $m_1, \ldots, m_t$ be pairs of germs from $(\mathbb{R}^r \times \mathbb{R}^n, 0)$ to $(\mathbb{R}^p \times \mathbb{R}^q, 0)$ and let $m_{i,0}$ be their respective restrictions to subspace $\{0\} \times \mathbb{R}^n$, $1 \le i \le t$. Then the following conditions are equivalent:*

i) $\left\langle \frac{\partial(f_{01}, f_{02})}{\partial x_1}, \ldots, \frac{\partial(f_{01}, f_{02})}{\partial x_n} \right\rangle_{\varepsilon_n} + [I_{f_{01}} \cdot \varepsilon_{n,p} \times \{0\}] + [\{0\} \times I_{f_{02}} \cdot \varepsilon_{n,q}] + \mathbb{R} \cdot \{m_{1,0}, \ldots, m_{t,0}\}$ $= \varepsilon_{n,p+q}$.

ii) $\left\langle \frac{\partial(f_1, f_2)}{\partial x_1}, \ldots, \frac{\partial(f_1, f_2)}{\partial x_n} \right\rangle_{\varepsilon_{r+n}} + [I_{f_1} \cdot \varepsilon_{r+n,p} \times \{0\}] + [\{0\} \times I_{f_2} \cdot \varepsilon_{r+n,q}] + \langle m_1, \ldots, m_t \rangle_{\varepsilon_r}$ $= \varepsilon_{r+n,p+q}$.

Now we are able to prove the Proposition 3.19.

Proof of Proposition 3.19.

Proof. Let (f_1, f_2) be a bi-$\mathcal{K}$-versal deformation of (f_{01}, f_{02}). We need to show that (f_1, f_2) is bi-$\mathcal{K}$-transversal.

In fact, first consider any germ $(g_{01}, g_{02}) \in \varepsilon_{n,p+q}$ and construct the following 1-parameter deformation (g_1, g_2) of (f_{01}, f_{02}):

$$(g_1, g_2)(t, x) = (f_{01}, f_{02})(x) + t(g_{01}, g_{02})(x).$$

By hypothesis, (g_1, g_2) is bi-$\mathcal{K}$-equivalent (as deformation) to induced deformation $\Gamma = \gamma^\star(f_1, f_2)$, $\Gamma = (\Gamma_1, \Gamma_2)$, with $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^r, 0)$, $\gamma = (\gamma_1, \ldots, \gamma_r)$.

Then,

$$\Gamma(t, x) = (f_1, f_2)(\gamma(t), x) = (f_1(\gamma(t), x), f_2(\gamma(t), x))$$

and

$$\dot{\Gamma} = \left(\frac{\partial \gamma_1}{\partial t}(0)\dot{f}_{11} + \cdots + \frac{\partial \gamma_r}{\partial t}(0)\dot{f}_{1r}, \frac{\partial \gamma_1}{\partial t}(0)\dot{f}_{21} + \cdots + \frac{\partial \gamma_r}{\partial t}(0)\dot{f}_{2r}\right) =$$
$$= \frac{\partial \gamma_1}{\partial t}(0)(\dot{f}_1, \dot{f}_2)_1 + \cdots + \frac{\partial \gamma_r}{\partial t}(0)(\dot{f}_1, \dot{f}_2)_r.$$

Hence $\dot{\Gamma} \in \mathbb{R} \cdot \{(\dot{f}_1, \dot{f}_2)_1, \ldots, (\dot{f}_1, \dot{f}_2)_r\}$.

Since (g_1, g_2) and (Γ_1, Γ_2) are bi-$\mathcal{K}$-equivalent, there exists a germ of diffeomorphism Ψ which is an unfolding of the identity of $\mathbb{R}^n$, $\Psi(t,x) = (t, \psi(t,x))$, with $\psi(0,x) = 0$, and such that

$$\Psi^\star(\gamma^\star(I_{f_1})) = I_{g_1} \quad \text{and} \quad \Psi^\star(\gamma^\star(I_{f_2})) = I_{g_2}.$$

Then, by Proposition 3.5 we can write

$$(\gamma^\star(f_1, f_2) \circ \Psi)(t,x) = M(t,x) \cdot (g_1, g_2)(t,x) = M(t,x) \cdot [(f_{01}, f_{02})(x) + t(g_{01}, g_{02})(x)]. \tag{3.1}$$

Since $M(0,x) = id_n(x)$, differentiating (3.1) with respect to t and evaluating in $t = 0$ we have

$$(\dot{g}_1, \dot{g}_2) - (\dot{\Gamma}_1, \dot{\Gamma}_2) \in TKK \cdot (f_{01}, f_{02}).$$

Hence,

$$(g_{01}, g_{02}) = (\dot{g}_1, \dot{g}_2) \in TKK \cdot (f_{01}, f_{02}) + \mathbb{R} \cdot \{(\dot{f}_1, \dot{f}_2)_1, \ldots, (\dot{f}_1, \dot{f}_2)_r\}.$$

Thus (f_1, f_2) is bi-$\mathcal{K}$-transversal.

On the other hand, let (f_1, f_2) be a bi-$\mathcal{K}$-transversal deformation. Let

$$(v,x) \mapsto (f_{01}, f_{02})(x) + (g_1, g_2)(v,x), \quad \text{with} \quad (g_1, g_2)(0,x) = 0$$

be any s-parameter deformation of (f_{01}, f_{02}).

Construct some deformation, with $(r+s)$-parameters, given by

$$(h_1, h_2)(u,v,x) = (f_1, f_2)(u,x) + (g_1, g_2)(v,x).$$

It is sufficient to show that the deformation (h_1, h_2) is bi-$\mathcal{K}$-equivalent to pull-back of (f_1, f_2) by a submersion $\bar{\gamma} : (\mathbb{R}^r \times \mathbb{R}^s, 0) \to (\mathbb{R}^r, 0)$. For this, it is sufficient to apply s-times the Reduction Lemma 3.21 and the Lemma 3.22.

In fact, first notice that $(h_1, h_2)|_{v=0} = (f_1, f_2)$. The bi-$\mathcal{K}$-transversality condition joint with the Lemma 3.22 provide:

$$\left\langle \frac{\partial(h_1,h_2)}{\partial x_1}, \ldots, \frac{\partial(h_1,h_2)}{\partial x_n} \right\rangle_{\varepsilon_{r+s+n}} + \{[I_{h_1} \cdot \varepsilon_{r+s+p,p} \times \{0\}] + [\{0\} \times I_{h_2} \cdot \varepsilon_{r+s+q,q}]\}$$

$$+ \left\langle \frac{\partial(h_1,h_2)}{\partial u_1}, \ldots, \frac{\partial(h_1,h_2)}{\partial u_r} \right\rangle_{\varepsilon_{r+s}} = \varepsilon_{r+s+n,p+q}. \tag{3.2}$$

Consider the initial velocity $\frac{\partial(h_1,h_2)}{\partial v_1}$ relative to deformation (h_1,h_2) with respect to parameter v_1. Then $\frac{\partial(h_1,h_2)}{\partial v_1} \in \varepsilon_{r+s+n,p+q}$. That is,

$$\frac{\partial h_1}{\partial v_1} \in \varepsilon_{r+s+n,p} \quad \text{and} \quad \frac{\partial h_2}{\partial v_1} \in \varepsilon_{r+s+n,q}.$$

Since $\varepsilon_{r+s+n,p+q}$ can be factored as (3.2), we can write

$$\frac{\partial(h_1,h_2)}{\partial v_1} = \sum_{j=1}^{n} X_j(u,v,x)\frac{\partial(h_1,h_2)}{\partial x_j} + (Y,Z)\circ h + \sum_{i=1}^{r}\xi_i(u,v)\frac{\partial(h_1,h_2)}{\partial u_i}$$

with $X_j \in \varepsilon_{r+s+n}$ and $\xi_i \in \varepsilon_{r+s}$.

By Reduction Lemma 3.21, there exists a germ of submersion $\bar{\gamma}_1 : (\mathbb{R}^r \times \mathbb{R}^s, 0) \to (\mathbb{R}^r \times \mathbb{R}^{s-1}, 0)$ such that (h_1,h_2) is bi-$\mathcal{K}$-equivalent to $\bar{\gamma}^\star(h_1,h_2)^1$, where $(h_1,h_2)^1 = (h_1,h_2)|_{v_1=0}$.

By recurrence over $v_2,\ldots,v_s$ we can obtain a germ of submersion $\bar{\gamma} : (\mathbb{R}^r \times \mathbb{R}^s, 0) \to (\mathbb{R}^r, 0)$ satisfying the required conditions to conclude the proof of Proposition 3.19. $\square$

§ 3.4. Bi-$\mathcal{A}$-trivial unfoldings of pairs

Definition 3.23. Let $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be a pair of germs. An *r-parameter unfolding of type* $(s, r-s)$ of (f_{01}, f_{02}) is a pair of germs $(F_1,F_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$ such that

$$(F_1,F_2)(u,x) = (u_1,\ldots,u_s, f_1(u,x), u_{s+1},\ldots,u_r, f_2(u,x)),$$

where (f_1,f_2) is an r-deformation of (f_{01}, f_{02}).

Definition 3.24. Two r-parameter unfoldings of type $(s, r-s)$ (F_1,F_2) and (G_1,G_2) of (f_{01}, f_{02}) are said to be *bi-$\mathcal{A}$-equivalent* if there exist germs of diffeomorphisms $\phi : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^r \times \mathbb{R}^n, 0)$, $\theta : (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$ where $\theta = \theta_1 \times \theta_2$; $\theta_1 : (\mathbb{R}^s \times \mathbb{R}^p, 0) \to (\mathbb{R}^s \times \mathbb{R}^p, 0)$ and $\theta_2 : (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0) \to (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$, for which the following diagram is commutative

$$\begin{array}{ccc}
(\mathbb{R}^r \times \mathbb{R}^n, 0) & \overset{(F_1,F_2)}{\longrightarrow} & (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{s-r} \times \mathbb{R}^q, 0) \\
\phi \downarrow & & \downarrow \theta = \theta_1 \times \theta_2 \\
(\mathbb{R}^r \times \mathbb{R}^n, 0) & \overset{(G_1,G_2)}{\longrightarrow} & (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{s-r} \times \mathbb{R}^q, 0)
\end{array}$$

and the germ ϕ has the form $\phi(u,x) = (u, \tilde{\phi}(u,x))$, $\tilde{\phi}(0,x) = x$.

Definition 3.25. An r-parameter unfolding of type $(s, r-s)$ (F_1,F_2) of (f_{01}, f_{02}) is said to be *bi-$\mathcal{A}$-trivial* if it is bi-$\mathcal{A}$-equivalent to the constant unfolding

$$(G_1,G_2)(u,x) = (u_1,\ldots,u_s, f_{01}(x), u_{s+1},\ldots,u_r, f_{02}(x)).$$

Definition 3.26. A pair of germs $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ is said to be *bi-stable* if all unfolding of type $(s, r-s)$ (F_1, F_2) of (f_{01}, f_{02}) is bi-$\mathcal{A}$-trivial.

§ 4. Bi-$\mathcal{A}$-classification

Let $(F_1, F_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q, 0)$ be a bi-stable pair of germs. We are interested in to describe the bi-$\mathcal{A}$-orbit of type $(s, r-s)$ of this pair. Without loss of generality, we will suppose that (F_1, F_2) has rank r (at 0) and F_1 has rank s (at 0). In this case, it is not difficult to show that (F_1, F_2) is bi-$\mathcal{A}$-equivalent to an unfolding of type $(s, r-s)$ of a pair $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ of rank 0 (at 0). Therefore, from now on we shall characterize the bi-$\mathcal{A}$-orbit of such pair via the bi-$\mathcal{K}$-equivalence of (f_{01}, f_{02}), where bi-$\mathcal{K}$-cod $(f_{01}, f_{02}) \leq r + p + q$.

Let $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be a regular r-deformation of $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$.

Denote $V_{f_1} = f_1^{-1}(0)$ and $V_{f_2} = f_2^{-1}(0)$, which are submanifolds of $\mathbb{R}^r \times \mathbb{R}^n$ of codimension p and q, respectively. For $s \leq r$, consider the germs of projections

$$\pi_1 : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s, 0) \quad \text{and} \quad \pi_2 : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^{r-s}, 0)$$

given by $\pi_1(u, x) = (u_1, \ldots, u_s)$ and $\pi_2(u, x) = (u_{s+1}, \ldots, u_r)$. Taking the restriction germs of these usual projections we define

$$\pi_{f_1} : (V_{f_1}, 0) \subset (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s, 0)$$

and

$$\pi_{f_2} : (V_{f_2}, 0) \subset (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^{r-s}, 0),$$

by $\pi_{f_1} = \pi_1|_{V_{f_1}}$ and $\pi_{f_2} = \pi_2|_{V_{f_2}}$.

Then it is well defined the pair:

$$(\pi_{f_1}, \pi_{f_2}) : (V_{f_1} \cap V_{f_2}, 0) \subset (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^{r-s}, 0)$$

given by $(\pi_{f_1}, \pi_{f_2})(u, x) = (u_1, \ldots, u_s, u_{s+1}, \ldots, u_r)$.

Next, we define the notion of bi-equivalence of type $(s, r-s)$ for two restriction germs (π_{f_1}, π_{f_2}) and (π_{g_1}, π_{g_2}) associated to regular deformations (f_1, f_2) and (g_1, g_2) of (f_{01}, f_{02}) and (g_{01}, g_{02}), respectively.

Definition 4.1. The restriction germs $(\pi_{f_1}, \pi_{f_2}) : (V_{f_1} \cap V_{f_2}, 0) \subset (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^{r-s}, 0)$ and $(\pi_{g_1}, \pi_{g_2}) : (V_{g_1} \cap V_{g_2}, 0) \subset (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^{r-s}, 0)$ are said to be *bi-equivalent of type* $(r, s-r)$ if there exists a germ of diffeomorphism $\Phi : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^{r-s} \times \mathbb{R}^n, 0)$ given by

$$\Phi(u, x) = (\gamma_1(u_1, \ldots, u_s), \gamma_2(u_{s+1}, \ldots, u_r), \psi(u, x))$$

such that $\Phi(V_{f_1}) = V_{g_1}$ and $\Phi(V_{f_2}) = V_{g_2}$, where $\gamma_1 : (\mathbb{R}^s, 0) \to (\mathbb{R}^s, 0)$ and $\gamma_2 : (\mathbb{R}^{r-s}, 0) \to (\mathbb{R}^{r-s}, 0)$ are germs of diffeomorphisms and the following diagrams are commutative:

$$\begin{array}{ccc} (V_{f_1}, 0) & \xrightarrow{\pi_{f_1}} & (\mathbb{R}^s, 0) \\ \Phi_1 \downarrow & & \downarrow \gamma_1 \\ (V_{g_1}, 0) & \xrightarrow{\pi_{g_1}} & (\mathbb{R}^s, 0) \end{array}$$

and

$$\begin{array}{ccc} (V_{f_2}, 0) & \xrightarrow{\pi_{f_2}} & (\mathbb{R}^{r-s}, 0) \\ \Phi_2 \downarrow & & \downarrow \gamma_2 \\ (V_{g_2}, 0) & \xrightarrow{\pi_{g_2}} & (\mathbb{R}^{r-s}, 0) \end{array}$$

where $\Phi_i = \Phi|_{V_{f_i}}$, $i = 1, 2$.

The next proposition characterizes the bi-equivalence for pairs of restriction germs when $r = p+q$ via the notion of bi-$\mathcal{K}$-isomorphism which involves a change of parameter of type $\gamma = \gamma_1 \times \gamma_2$ (cartesian product).

Proposition 4.2. *([21]) Let $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be regular $(p+q)$-deformations of $(f_{01}, f_{02}), (g_{01}, g_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$, respectively. Then, (π_{f_1}, π_{f_2}) and (π_{g_1}, π_{g_2}) are bi-equivalent if and only if (f_1, f_2) and (g_1, g_2) are bi-$\mathcal{K}$-isomorphic via a change of parameters $\gamma : (\mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$,$\gamma = \gamma_1 \times \gamma_2$.*

Now, let $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ be any r-deformation of $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ (not necessarily regular). Consider the regular extension of (f_1, f_2) given by $(\bar{f}_1, \bar{f}_2) : (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$

$$(u_1, \ldots, u_s, y, u_{s+1}, \ldots, u_r, z, x) \mapsto (-y + f_1(u, x), -z + f_2(u, x)),$$

where $u = (u_1, \ldots, u_s, u_{s+1}, \ldots, u_r)$.

Since $(\bar{f}_1, \bar{f}_2)$ is regular, the respective restriction germ associated to it is given by

$$(\pi_{\bar{f}_1}, \pi_{\bar{f}_2}) : (V_{\bar{f}_1} \cap V_{\bar{f}_2}, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$$

$$(u_1, \ldots, u_s, y, u_{s+1}, \ldots, u_r, z, x) \mapsto (u_1, \ldots, u_s, y, u_{s+1}, \ldots, u_r, z)$$

which can be identified as an unfolding of type $(s, r-s)$ of (f_{01}, f_{02}) given by

$$(F_1, F_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0),$$

$$(F_1, F_2)(u, x) = (u_1, \ldots, u_s, f_1(u, x), u_{s+1}, \ldots, u_r, f_2(u, x)),$$

where $\pi_{\bar{f}_1} = \pi_1|_{V_{\bar{f}_1}}$, $\pi_{\bar{f}_2} = \pi_2|_{V_{\bar{f}_2}}$ and $\pi_1 : (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^p, 0)$ and $\pi_2 : (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q \times \mathbb{R}^n, 0) \to (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$ are the canonical projections.

Also, notice that $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2})$ is an unfolding of type (p, q) of (π_{f_1}, π_{f_2}).

Proposition 4.3. *Let* (F_1, F_2), $(G_1, G_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$ *be unfoldings of type* $(s, r-s)$ *of* (f_{01}, f_{02}), $(g_{01}, g_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$, *respectively. If* (F_1, F_2) *and* (G_1, G_2) *are bi-*$\mathcal{A}$*-equivalent then* (f_{01}, f_{02}) *and* (g_{01}, g_{02}) *are bi-*$\mathcal{K}$*-equivalent.*

Proof. Let

$$(F_1, F_2)(u, x) = (u_1, \ldots, u_s, f_1(u, x), u_{s+1}, \ldots, u_r, f_2(u, x))$$

and

$$(G_1, G_2)(u, x) = (u_1, \ldots, u_s, g_1(u, x), u_{s+1}, \ldots, u_r, g_2(u, x)).$$

Using the previous identification

$$(\pi_{\bar{f}_1}, \pi_{\bar{f}_2}) \equiv (F_1, F_2) \quad \text{and} \quad (\pi_{\bar{g}_1}, \pi_{\bar{g}_2}) \equiv (G_1, G_2),$$

it follows from hypothesis that $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2})$ and $(\pi_{\bar{g}_1}, \pi_{\bar{g}_2})$ are bi-equivalent of type $(s, r-s)$. Then there exists a germ of diffeomorphism $\Phi : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0)$ such that the following diagrams are commutative:

$$\begin{array}{ccc} (V_{\bar{f}_1}, 0) & \xrightarrow{\pi_{\bar{f}_1}} & (\mathbb{R}^s \times \mathbb{R}^p, 0) \\ \Phi_1 \downarrow & & \downarrow \gamma_1 \\ (V_{\bar{g}_1}, 0) & \xrightarrow{\pi_{\bar{g}_1}} & (\mathbb{R}^s \times \mathbb{R}^p, 0) \end{array}$$

and

$$\begin{array}{ccc} (V_{\bar{f}_2}, 0) & \xrightarrow{\pi_{\bar{f}_2}} & (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0) \\ \Phi_2 \downarrow & & \downarrow \gamma_2 \\ (V_{\bar{g}_2}, 0) & \xrightarrow{\pi_{\bar{g}_2}} & (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0) \end{array}$$

where

$$\Phi(y, z, u, x) = (\gamma_1(u_1, \ldots, u_s, y), \gamma_2(u_{s+1}, \ldots, u_r, z), \psi(y, z, u, x)),$$

$u = (u_1, \ldots, u_s, u_{s+1}, \ldots, u_r)$ with

$$\gamma_1 : (\mathbb{R}^s \times \mathbb{R}^p, 0) \to (\mathbb{R}^s \times \mathbb{R}^p, 0), \;\; \gamma_2 : (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0) \to (\mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$$

germs of diffeomorphisms, $\Phi_1 = \Phi|_{V_{\bar{f}_1}}$ and $\Phi_2 = \Phi|_{V_{\bar{f}_2}}$.

Since $\bar{f}_1, \bar{g}_1$ and $\bar{f}_2, \bar{g}_2$ are regular deformations and $\Phi(V_{\bar{f}_i}) = V_{\bar{g}_i}$, $i = 1, 2$, we have that $(\bar{g}_1, \bar{g}_2) \circ \Phi$ is bi-$\mathcal{C}$-equivalent to $(\bar{f}_1, \bar{f}_2)$ (see [14, p. 27]). Thus, by Proposition 3.5, $(\bar{g}_1, \bar{g}_2) \circ \Phi$ can be write as

$$(\bar{g}_1, \bar{g}_2) \circ \Phi = (M_1 \cdot \bar{f}_1, M_2 \cdot \bar{f}_2),$$

where M_1, M_2 are invertible matrices.

Taking the restriction to subspace $\{y = 0, z = 0, u = 0\}$ we obtain a germ of diffeomorphism $k : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

$$(g_{01}(k(x)), g_{02}(k(x))) = (M_{01}(x) \cdot f_{01}(x), M_{02}(x) \cdot f_{02}(x)).$$

In other words, (g_{01}, g_{02}) is bi-$\mathcal{K}$-equivalent to (f_{01}, f_{02}). $\square$

Proposition 4.4. *([21]) Let $(\bar{f}_1, \bar{f}_2) : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ as previously defined. Then $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2})$ is an trivial unfolding of (π_{f_1}, π_{f_2}) if and only if $(\bar{f}_1, \bar{f}_2)$ is bi-$\mathcal{K}$-equivalent to $\gamma^\star(f_1, f_2)$, where $\gamma : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ is a submersion germ given by $\gamma(y, z, u) = (\gamma_1(y, u), \gamma_2(z, u))$.*

Proposition 4.5. *Let $(\bar{f}_1, \bar{f}_2) : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ as previously defined. Then, $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2})$ is bi-stable if and only if $(\bar{f}_1, \bar{f}_2)$ is a bi-$\mathcal{K}$-versal deformation.*

Proof. This proof is an adaptation of a similar result given in [21, Proposition 3.3] (p. 68) to our case. Notice that here $(\bar{f}_1, \bar{f}_2)$ is a regular $(p + q)$-deformation of $(f_1, f_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$.

We just will show the necessary condition. The complete proof does not use techniques lying outside the scope of this paper. However a careful version would occupy more space than is available.

Let $(\hat{g}_1, \hat{g}_2) : (\mathbb{R}^t \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ any t-parameter deformation of (f_1, f_2). Without loss of generality, we can write it as following:

$$(\hat{g}_1, \hat{g}_2)(w, u, x) = (f_1, f_2)(u, x) + (g_1, g_2)(w, u, x)$$

where $(g_1, g_2)(0, u, x) = (0, 0)$.

Then, the $(t + p + q)$-deformation of (f_1, f_2), given by

$$(\hat{f}_1, \hat{f}_2) : (\mathbb{R}^t \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$$

$$(w, y, z, u, x) \mapsto (\bar{f}_1, \bar{f}_2)(y, z, u, x) + (g_1, g_2)(w, u, x)$$

is a regular extension of $(\bar{f}_1, \bar{f}_2)$.

Hence the pair

$$(\pi_{\hat{f}_1}, \pi_{\hat{f}_2}) : (V_{\hat{f}_1} \cap V_{\hat{f}_2}, 0) \subset (\mathbb{R}^t \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^t \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r, 0)$$

$$(w, y, z, u, x) \mapsto (w, y, z, u)$$

is a t-unfolding of

$$(\pi_{\bar{f}_1}, \pi_{\bar{f}_2}) : (V_{\bar{f}_1} \cap V_{\bar{f}_2}, 0) \subset (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r, 0).$$

From hypothesis of bi-stability of $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2})$, we have that $(\pi_{\hat{f}_1}, \pi_{\hat{f}_2})$ is trivial. Then, by Proposition 4.4, $(\hat{f}_1, \hat{f}_2)$ is bi-$\mathcal{K}$-equivalent to $\gamma^\star(\bar{f}_1, \bar{f}_2)$ where $\gamma : (\mathbb{R}^t \times \mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ is a germ of submersion, with $\gamma(w, y, z) = (\gamma_1(w, y), \gamma_2(w, z))$. By Proposition 3.5 we can write

$$\hat{f}_1 = M_1 \cdot (\gamma^\star \bar{f}_1 \circ \phi)$$

and

$$\hat{f}_2 = M_2 \cdot (\gamma^\star \bar{f}_2 \circ \phi),$$

where M_1 and M_2 are invertible matrices and $\phi : (\mathbb{R}^t \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^t \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^n, 0)$ is a diffeomorphism, $\phi(w, y, z, u, x) = (w, y, z, \bar{\phi}(w, y, z, u, x))$.

Taking the restriction to subspace $\{y = 0, z = 0\}$, we obtain

$$(\hat{f}_1, \hat{f}_2)(w, 0, 0, u, x) = (\bar{f}_1, \bar{f}_2)(0, 0, u, x) + (g_1, g_2)(w, u, x)$$

$$= (f_1, f_2)(u, x) + (g_1, g_2)(w, u, x) = (\hat{g}_1, \hat{g}_2)(w, u, x).$$

This analysis permit us to conclude that $(\hat{g}_1, \hat{g}_2)$ is bi-$\mathcal{K}$-equivalent to $\gamma_o^\star(\bar{f}_1, \bar{f}_2)$, where γ_o is the germ given by the restriction of γ to subspace $\{y = 0, z = 0\}$ (i.e., $\gamma_o : (\mathbb{R}^t, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$, $\gamma_o(w) = \gamma(w, 0, 0)$).

□

The next proposition gives the converse of the Proposition 4.3.

Proposition 4.6. *Let $(F_1, F_2), (G_1, G_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$ be two bi-stable unfoldings of $(f_{01}, f_{02}), (g_{01}, g_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$, respectively. If (f_{01}, f_{02}) and (g_{01}, g_{02}) are bi-$\mathcal{K}$-equivalent, then (F_1, F_2) and (G_1, G_2) are bi-$\mathcal{A}$-equivalent.*

Proof. Consider the $(p+q)$-regular deformations $(\bar{f}_1, \bar{f}_2)$ and $(\bar{g}_1, \bar{g}_2)$ of (f_1, f_2) and (g_1, g_2), respectively, where (f_1, f_2),(g_1, g_2) are r-deformations of $(f_{01}, f_{02}), (g_{01}, g_{02})$, as previously defined. Then we may to obtain the following identifications:

$$(F_1, F_2) \equiv (\pi_{\bar{f}_1}, \pi_{\bar{f}_2}) \quad \text{and} \quad (G_1, G_2) \equiv (\pi_{\bar{g}_1}, \pi_{\bar{g}_2}).$$

From bi-stability of hypothesis and by Proposition 4.5, follows that $(\bar{f}_1, \bar{f}_2)$ and $(\bar{g}_1, \bar{g}_2)$ are bi-$\mathcal{K}$-versal deformations.

Let $(\hat{g}_1, \hat{g}_2)$ be the $(p+q)$-parameter bi-$\mathcal{K}$-versal deformation of (g_{01}, g_{02}) given by

$$(\hat{g}_1, \hat{g}_2)(y, z, x) = (M_1(x) \cdot \bar{f}_1(y, z, \phi(x)), M_2(x) \cdot \bar{f}_2(y, z, \phi(x))),$$

where M_1, M_2 and ϕ are, respectively, invertible matrices and a germ of diffeomorphism $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ whose existence is guaranteed by bi-$\mathcal{K}$-equivalence between (f_{01}, f_{02}) and (g_{01}, g_{02}), i.e.,

$$(g_{01}, g_{02}) = (M_1 \cdot f_{01} \circ \phi, M_2 \cdot f_{02} \circ \phi).$$

Then $(\hat{g}_1, \hat{g}_2)$ is bi-$\mathcal{K}$-equivalent to $\gamma^\star(\bar{g}_1, \bar{g}_2)$, where $\gamma = \gamma_1 \times \gamma_2 : (\mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$.

Since $(\hat{g}_1, \hat{g}_2)$ and $(\bar{g}_1, \bar{g}_2)$ are bi-$\mathcal{K}$-versal deformations of the same pair, γ is a germ of diffeomorphism. Hence, by Proposition 4.2, $(\pi_{\hat{g}_1}, \pi_{\hat{g}_2})$ and $(\pi_{\bar{g}_1}, \pi_{\bar{g}_2})$ are bi-equivalent.

On the other hand, consider the following germ of diffeomorphism

$$\Phi = id_p \times id_q \times \phi : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^n, 0).$$

It is easy to check that for all $(y, z, x) \in V_{\hat{g}_i}$ then $\Phi(y, z, x) \in V_{\bar{f}_i}$, $i = 1, 2$.

Then we may obtain a bi-equivalence between $(\pi_{\hat{g}_1}, \pi_{\hat{g}_2})$ and $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2})$.

By transitivity and using the previous identifications again, follows that (F_1, F_2) and (G_1, G_2) are bi-$\mathcal{A}$-equivalent. □

Finally, let S be the set of all classes of $(s, r-s)$ bi-$\mathcal{A}$-equivalence of pairs of germs $(F_1, F_2) : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^s \times \mathbb{R}^p \times \mathbb{R}^{r-s} \times \mathbb{R}^q, 0)$ which are bi-stable, $\mathrm{rank}_0(F_1, F_2) = r$ and $\mathrm{rank}_0 F_1 = s$.

Let K be the set of all classes of bi-$\mathcal{K}$-equivalence of pairs from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^p \times \mathbb{R}^q, 0)$ of rank 0 and bi-$\mathcal{K}$-codimension $\leq p + q + r$.

Theorem 4.7. *The map Ψ given by $(F_1, F_2) \mapsto ({f_1}_{F_1}, {f_2}_{F_2})$ induces a bijection between the sets S and K. (Here ${f_i}_{F_i} : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ indicates the unique germ of rank 0 which F_i unfolds, $i = 1, 2$).*

Proof. Observe that $\Psi(F_1, F_2)$ and $\Psi(G_1, G_2)$ belong to same class of bi-$\mathcal{K}$-equivalence. Then, by Proposition 4.6, $(F_1, F_2), (G_1, G_2)$ belong to same class in S, i.e., they are $(s, r-s)$ bi-$\mathcal{A}$-equivalent.

On the other hand, considering the pair $(f_{01}, f_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ with rank 0 and bi-$\mathcal{K}$-codimension $\leq r + p + q$, we may construct the following regular $(p + q + r)$-deformation, bi-$\mathcal{K}$-transversal (and hence bi-$\mathcal{K}$-versal)

$$(\bar{f}_1, \bar{f}_2)(y, z, u, x) = (-y + f_1(u, x), -z + f_2(u, x)),$$

where (f_1, f_2) is an r-deformation of (f_{01}, f_{02}). Applying the Proposition 4.5, follows that $(\pi_{\bar{f}_1}, \pi_{\bar{f}_2}) \equiv (F_1, F_2)$ is bi-stable. Then we conclude that induced map by Ψ is surjective and injective.

Notice that the map between S and K, induced by Ψ, is well defined.

Let (F_1, F_2) and $(G_1, G_2) \in S$, pairs bi-$\mathcal{A}$-equivalent of type $(s, r-s)$. Then $\Psi(F_1, F_2)$ and $\Psi(G_1, G_2)$ are bi-$\mathcal{K}$-equivalent. In fact, it is sufficient consider the identifications

$$(\pi_{\bar{f}_1}, \pi_{\bar{f}_2}) \equiv (F_1, F_2) \quad \text{and} \quad (\pi_{\bar{g}_1}, \pi_{\bar{g}_2}) \equiv (G_1, G_2).$$

Then (f_{01}, f_{02}) and (g_{01}, g_{02}) are bi-$\mathcal{K}$-equivalent, as showed in Proposition 4.6. □

Thus the problem of classifying bi-stable pairs of germs under $(s, r-s)$ bi-$\mathcal{A}$-equivalence reduces to the problem of classifying pairs of germs under the relation of bi-$\mathcal{K}$-equivalence, up to certain bi-$\mathcal{K}$-codimension.

As a consequence, we also recover the version for pairs of classical Mather's classification theorem (Theorem 2.4):

Theorem 4.8. *([21]) If the pairs of germs $(f_{01}, f_{02}), (g_{01}, g_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-stable then they are bi-$\mathcal{A}$-equivalent if and only if they are bi-$\mathcal{K}$-equivalent.*

Moreover, in [21], the author introduces the notion of coherent isomorphism which means "induced by a same isomorphism". With the notion of coherent isomorphism the Theorem 4.8 can be reformulated as following:

Theorem 4.9. *([23],[21]) If the pairs of germs $(f_{01}, f_{02}), (g_{01}, g_{02}) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-stable then they are bi-$\mathcal{A}$-equivalent if and only the $\mathbb{R}$-algebras $Q(f_{01})$, $Q(g_{01})$ and $Q(f_{02}), Q(g_{02})$ are isomorphic by coherent isomorphisms.*

§ 5. Topological bi-$\mathcal{K}$-equivalence of pairs

In this Section we introduce the notion of topological bi-$\mathcal{K}$-equivalence, showing examples, properties and also proposing some open questions about this new equivalence relation.

To investigate weaker versions of classical equivalence relations is a subject studied by several authors. For instance, with respect to topological $\mathcal{K}$-equivalence we can cite [17], [18], [1], [4], [5], [6]. In a same way, it seems natural to investigate the topological version of bi-$\mathcal{K}$-equivalence and its relation with topological bi-$\mathcal{A}$-equivalence for pairs of germs.

Definition 5.1.

i) Two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be *topologically $\mathcal{A}$-equivalent* (or C^0-$\mathcal{A}$-equivalent) if there exist germs of homeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that

$$g = k \circ f \circ h.$$

ii) Two smooth pairs of germs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are said to be *topologically bi-$\mathcal{A}$-equivalent* (or bi-C^0-$\mathcal{A}$-equivalent) if there exist germs of homeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $k_1 : (\mathbb{R}^p \to (\mathbb{R}^p, 0)$ and $k_2 : (\mathbb{R}^q, 0) \to (\mathbb{R}^q, 0)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{R}^n, 0) & \stackrel{(f_1, f_2)}{\longrightarrow} & (\mathbb{R}^p \times \mathbb{R}^q, 0) \\ h \downarrow & & \downarrow k_1 \times k_2 \\ (\mathbb{R}^n, 0) & \stackrel{(g_1, g_2)}{\longrightarrow} & (\mathbb{R}^p \times \mathbb{R}^q, 0) \end{array}$$

iii) Two smooth map germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are said to be *topologically $\mathcal{K}$-equivalent* (or C^0-$\mathcal{K}$-equivalent) if there exist germs of homeomorphisms

$$H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0) \quad \text{and} \quad h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

such that the same properties in Definition 2.1 are satisfied. When $h = id_n$ we say that f and g are C^0-$\mathcal{C}$-equivalent.

iv) Two smooth pairs of germs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are said to be *bi-C^0-$\mathcal{K}$-equivalent* if there exist germs of homeomorphisms

$$H : (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q, 0) \quad \text{and} \quad h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$$

such that the same properties in Definition 3.2 are satisfied.

When $h = id_n$, we say that (f_1, f_2) and (g_1, g_2) are bi-C^0-$\mathcal{C}$-equivalent.

Example 5.2. The function germs $f(x) = x^3$ and $g(x) = x$ are C^0-$\mathcal{C}$-equivalent. It is enough considering $h = id_1 : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ and $H : (\mathbb{R} \times \mathbb{R}, (0, 0)) \longrightarrow (\mathbb{R} \times \mathbb{R}, (0, 0))$ given by $H(x, y) = (x, y^3)$. Notice these germs are not $\mathcal{C}$-equivalent because $I_f \neq I_g$.

Example 5.3. The germs $f(x) = (x^2, 0)$ and $g(x) = (x, 0)$ are $\mathcal{C}^0$-$\mathcal{K}$-equivalent (see [5, 6]). However, x and x^2 are not C^0-$\mathcal{K}$-equivalent as function germs in one variable (see [17, 6]).

Example 5.4. The pairs of germs $f = (f_1, f_2),\ g = (g_1, g_2) : (\mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$ given by

$$f(x) = (0, x) \quad \text{and} \quad g(x) = (x^2, 0)$$

are not bi-C^0-$\mathcal{K}$-equivalent.

In fact, suppose by contradiction that they are bi-C^0-$\mathcal{K}$-equivalent. Then there exist germs of homeomorphisms $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $H : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0)$, $H(x, y, z) = (h(x), H_1(x, y), H_2(x, z))$, satisfying the conditions of Definition 3.2. Hence, $(h, \hat{H}_1)$, $(h, \hat{H}_2)$ with $\hat{H}_i = (h, H_i)$, $i = 1, 2$ give C^0-$\mathcal{K}$-equivalences for f_1, g_1 and f_2, g_2, respectively. In other words, $\hat{H}_i(x, 0) = (h(x), H_i(x, 0)) = (h(x), 0)$ and $\hat{H}_i \circ (id_1, f_i) = (id_1, g_i) \circ h$, $i = 1, 2$.

Hence, $\hat{H}_1(x, 0) = \hat{H}_1 \circ (id_1, f_1)(x) = (id_1, g_1) \circ h(x) = (h(x), h(x)^2)$. Since by definition $\hat{H}_1(x, 0) = (h(x), 0)$ we have an absurdity. However, the germs $f(x) = (0, x)$ and $g(x) = (x^2, 0)$ are $\mathcal{C}^0$-$\mathcal{K}$-equivalent (Example 5.3).

Example 5.5. The pairs of germs $(f_1, f_2),\ (g_1, g_2) : (\mathbb{R}^2, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$ given by

$$f(x, y) = (x, y) \quad \text{and} \quad g(x, y) = (x, y^2)$$

are not bi-$\mathcal{C}^0$-$\mathcal{K}$-equivalent.

In fact the germs $f = (f_1, f_2)$ and $g = (g_1, g_2)$ have different topological degree which is an invariant for $\mathcal{C}^0$-$\mathcal{K}$-equivalence ([17], [6]). Then f and g are not $\mathcal{C}^0$-$\mathcal{K}$-equivalent. Consequently, they are not bi-$\mathcal{C}^0$-$\mathcal{K}$-equivalent.

Proposition 5.6. *The pairs of germs $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ are bi-C^0-$\mathcal{K}$-equivalent if and only if there exists a germ of homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that (f_1, f_2) and $(g_1, g_2) \circ h$ are bi-C^0-$\mathcal{C}$-equivalent.*

Proof. If (f_1, f_2) is bi-C^0-$\mathcal{K}$-equivalent to (g_1, g_2) then there exist homeomorphisms h and H, such that $H(x, y, z) = (h(x), H_1(x, y), H_2(x, z))$, $H(x, 0, 0) = (h(x), 0, 0)$ and $H \circ (id_n, f_1, f_2) = (id_n, g_1, g_2) \circ h$. Let $\mathcal{H}$ be the homeomorphism given by $\mathcal{H} = (h^{-1} \times id_p \times id_q) \circ H$. Via $(id_n, \mathcal{H})$, follows that (f_1, f_2) and $(g_1, g_2) \circ h$ are bi-C^0-$\mathcal{C}$-equivalent.

Reciprocally, let $(id_n, \mathcal{H})$, $\mathcal{H} = (id_n, \mathcal{H}_1, \mathcal{H}_2)$ be the bi-C^0-$\mathcal{C}$-equivalence between (f_1, f_2) and $(g_1, g_2) \circ h$. Consider the homeomorphism $H = (h, \mathcal{H}_1, \mathcal{H}_2)$. Via (h, H), follows that (f_1, f_2) and (g_1, g_2) are bi-C^0-$\mathcal{K}$-equivalent. $\square$

Remark. Are valid similar conditions that to the ones given in Remark after Definition 3.2. Just replace the words "bi-$\mathcal{K}$, $\mathcal{K}$ and bi-$\mathcal{A}$" respectively by "bi-C^0-$\mathcal{K}$, C^0-$\mathcal{K}$ and bi-C^0-$\mathcal{A}$".

The next proposition is an adapted version for pair of germs of a Nishimura's result (see [18, Theorem 1] (p. 83)):

Proposition 5.7. *Let $f_1, g_1 : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ and $f_2, g_2 : (\mathbb{R}^q, 0) \to (\mathbb{R}^q, 0)$ be finitely C^0-$\mathcal{K}$-determined map germs. Consider $(f_1, f_2), (g_1, g_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$, $n = p + q$ such that $(f_1, f_2)(y, z) = (f_1(y), f_2(z))$, $(g_1, g_2)(y, z) = (g_1(y), g_2(z))$, $(y, z) \in (\mathbb{R}^p \times \mathbb{R}^q, 0)$. Then, (f_1, f_2) and (g_1, g_2) are bi-C^0-$\mathcal{K}$-equivalent if and only if $|deg(f_i)| = |deg(g_i)|$, $i = 1, 2$, where deg means the mapping degree.*

Proof. Notice that follows from hypothesis of finitely determinacy that

$$f_1^{-1}(0) = g_1^{-1}(0) = \{0\} \quad \text{and} \quad f_2^{-1}(0) = g_2^{-1}(0) = \{0\}$$

as germs. Therefore, the mappings degree $deg(f_i)$ and $deg(g_i)$ are well defined, $i = 1, 2$.

If (f_1, f_2) and (g_1, g_2) are bi-C^0-$\mathcal{K}$-equivalent then f_1 and g_1 are C^0-$\mathcal{K}$-equivalent and also f_2 and g_2 are C^0-$\mathcal{K}$-equivalent. Then, by Nishimura's Theorem (cf. [18]), $|deg(f_i)| = |deg(g_i)|$, $i = 1, 2$.

On the other hand, the condition $|deg(f_i)| = |deg(g_i)|$, $i = 1, 2$ is equivalent to say that f_1 and g_1 are C^0-$\mathcal{K}$-equivalent and also f_2 and g_2 are C^0-$\mathcal{K}$-equivalent (again by

Nishimura's Theorem). Then, there exist germs of homeomorphisms

$$h_1 : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0), \quad H_1 : (\mathbb{R}^p \times \mathbb{R}^p, 0) \to (\mathbb{R}^p \times \mathbb{R}^p, 0),$$

$$h_2 : (\mathbb{R}^q, 0) \to (\mathbb{R}^q, 0), \quad H_2 : (\mathbb{R}^q \times \mathbb{R}^q, 0) \to (\mathbb{R}^q \times \mathbb{R}^q, 0)$$

such that

$$H_1(x,t) = (h_1(x), \theta_1(x,t)), H_2(y,w) = (h_2(y), \theta_2(y,w)), \theta_1(x,0) = 0, \theta_2(y,0) = 0,$$

$$H_1 \circ (id_p, f_1) = (id_p, g_1) \circ h_1 \quad \text{and} \quad H_2 \circ (id_q, f_2) = (id_q, g_2) \circ h_2.$$

Consider $h : (\mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^p \times \mathbb{R}^q, 0)$ given by $h(x,y) = (h_1(x), h_2(y))$ and

$$H : (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^q, 0) \to (\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^q, 0)$$

given by $H(x,y,t,w) = (h_1(x), h_2(y), \theta_1(x,t), \theta_2(y,w))$. Clearly h and H are homeomorphisms and they give the bi-C^0-$\mathcal{K}$-equivalence between (f_1, f_2) and (g_1, g_2). $\square$

For instance, the Proposition 5.7 permits concluding that the pair of germs given in Example 5.5 are not bi-C^0-$\mathcal{K}$-equivalent.

§ 5.1. Open questions

The bi-C^0-$\mathcal{K}$-equivalence introduced here is an interesting subject and many open questions about it can be formulated. For instance:

1) Is it possible to obtain a topological version of Mather's classification theorem, similar to Theorem 4.8?

2) How to obtain an invariant for bi-C^0-$\mathcal{K}$-equivalence, similar that given in Proposition 5.7, to larger class of germs?

3) Within a $\mathcal{K}$-orbit (or a bi-$\mathcal{K}$-orbit) what do say about the bi-C^0-$\mathcal{K}$-orbits?

4) How to obtain new invariants, normal forms and classifications for bi-C^0-$\mathcal{K}$-equivalence?

For instance, the Open Question 1) is being investigated in [7]. The Open Question 2) is treated in [8].

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